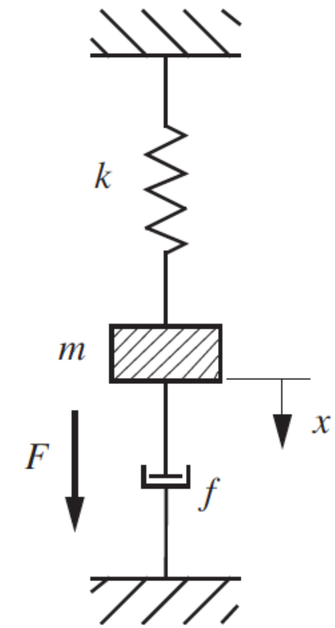
Week 03: Mathematical Modeling Part 2

Mahmut Selman Sakar

Institute of Mechanical Engineering, EPFL

Recap

- Modeling Process
- Mathematical Modeling of
  - Mechanical Systems
  - Electrical Circuits
  - Analogous Systems



Lecture Overview

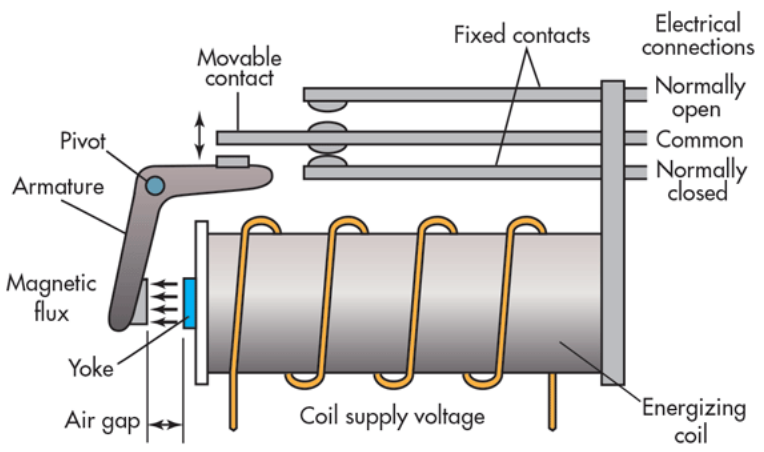
- Mathematical Modeling of Electromechanical Systems
- Lagrangian Formulation
- Next week: Linearization and State-space Representation

Electromechanical Systems

- Devices that carry out electrical operations by moving parts
  - Manually operated switch, generator, microphone

- Devices that involve an electrical signal to create mechanical movement
  - Relays, AC/DC motors, clocks, loudspeakers

- Piezoelectric materials (work in both ways)



Electromagnetic Induction and DC servomotor

- **DC servomotor:** A machine that converts electrical energy into rotation
  - stator and rotor

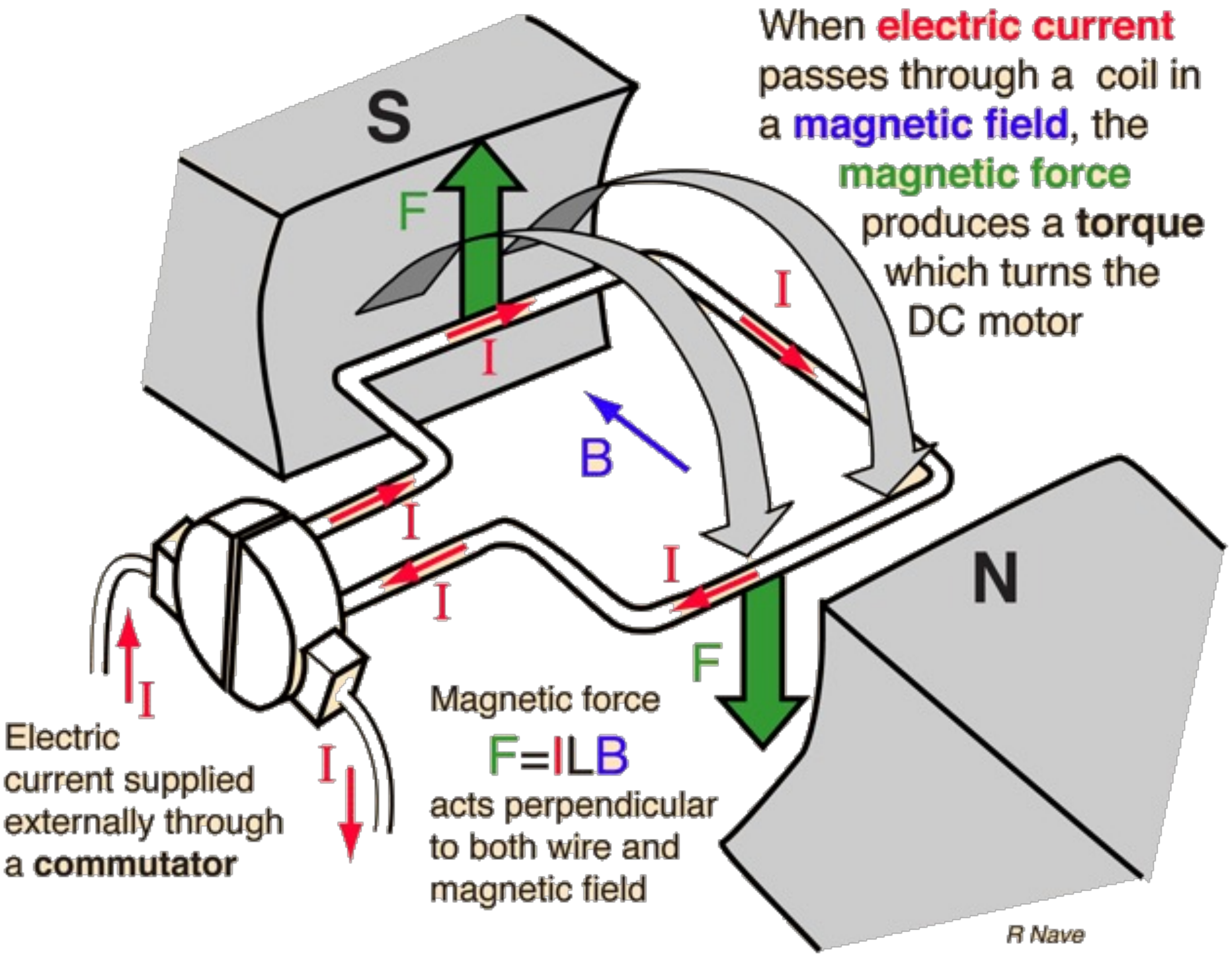
- Excitation (stator)
  - Permanent magnets generate the magnetic field: magneto
  - Electromagnetic coils generate the magnetic field: dynamo

- Rotor consists of armature winding

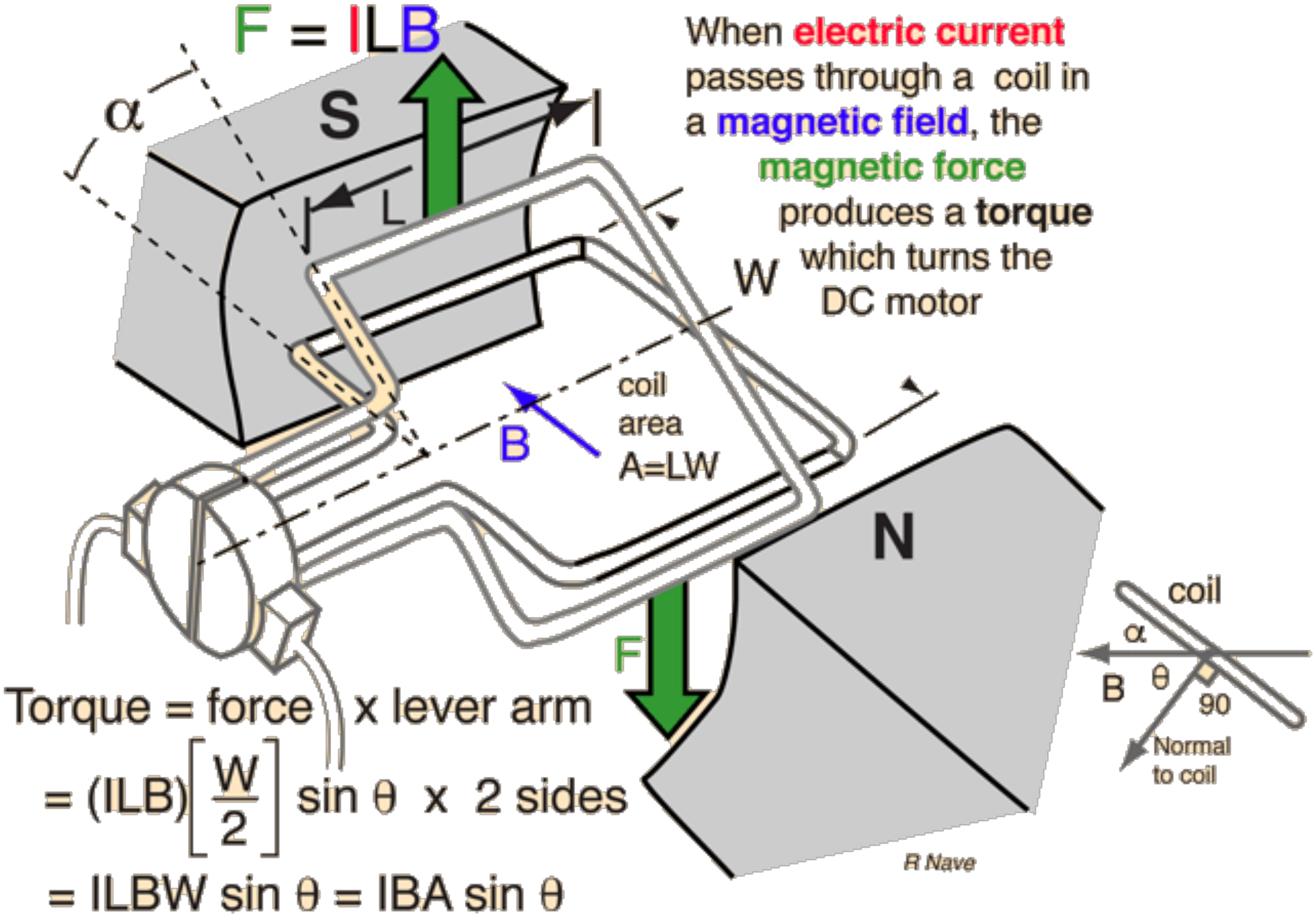
- **Armature control**: The field must be kept constant
  - Either the stator current is constant, or the stator coils are replaced by permanent magnets

video

DC Servomotor (Lorentz Law)



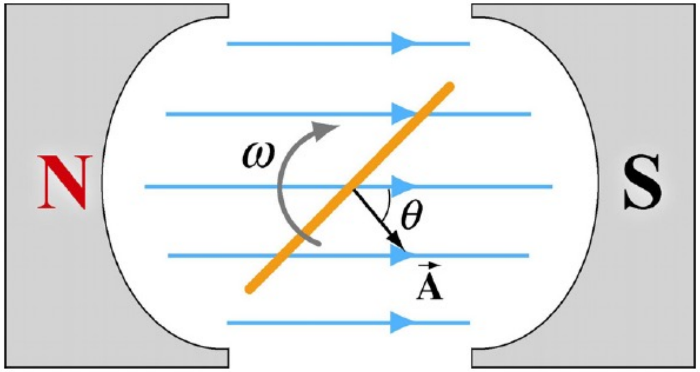
DC Servomotor



Electromotive Force

- **Faraday's Law of Induction**: The time derivative of the magnetic flux through a closed circuit induces an electromotive force in the circuit, which in turn drives a current.
- The electromotive force (emf) around a closed path is equal to the negative of the time rate of change of the magnetic flux enclosed by the path.

- **Lenz's Law**: The induced current produces magnetic fields which tend to oppose the change in flux that induces such currents.
- Analogous to Newton's third Law



$$\varepsilon = -\frac{d\Phi_B}{dt}$$

Elements of Electromechanical Systems

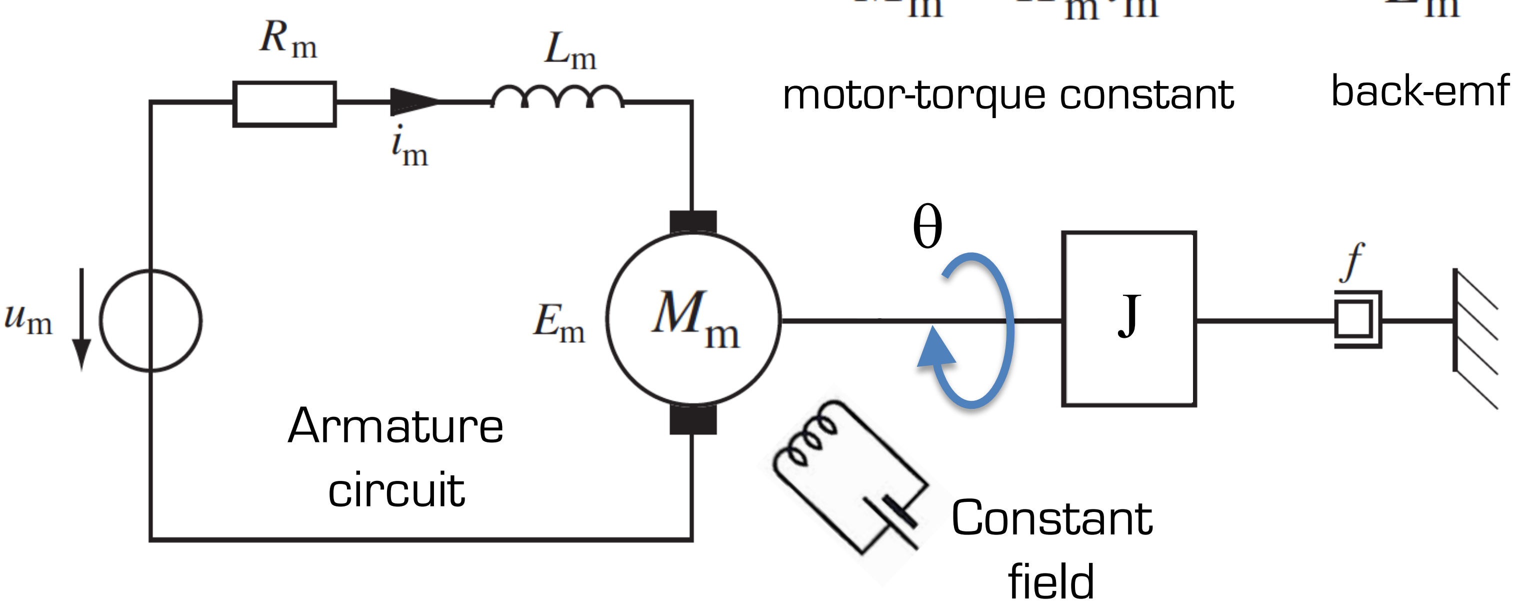
- Constant magnetic field: permanent magnets or constant current
  - Torque M_m becomes directly proportional to the armature current
  - The induced voltage E_m is directly proportional to the angular velocity

$$M_m = K_m i_m$$

motor-torque constant

$$E_m = K_m \dot{\theta}$$

back-emf constant



$$K_m : \left[\frac{\text{V s}}{(\text{rad})}\right]$$

Governing equations

- Kirchhoff's Law

$$i_m R_m + L_m \frac{di_m}{dt} + E_m - u_m = 0 \qquad (1)$$

- Newton's Law

$$J\dot{\omega} = K_m i_m - f\omega \quad \text{where} \quad \omega = \dot{\theta} \qquad (2)$$

By taking time derivative of (2) and combining with (1), we obtain:

$$J\ddot{\omega} = K_m \frac{1}{L_m}\left[u_m - \frac{1}{K_m}(J\dot{\omega} + f\omega)R_m - K_m\omega\right] - f\dot{\omega}$$

$$\Downarrow$$

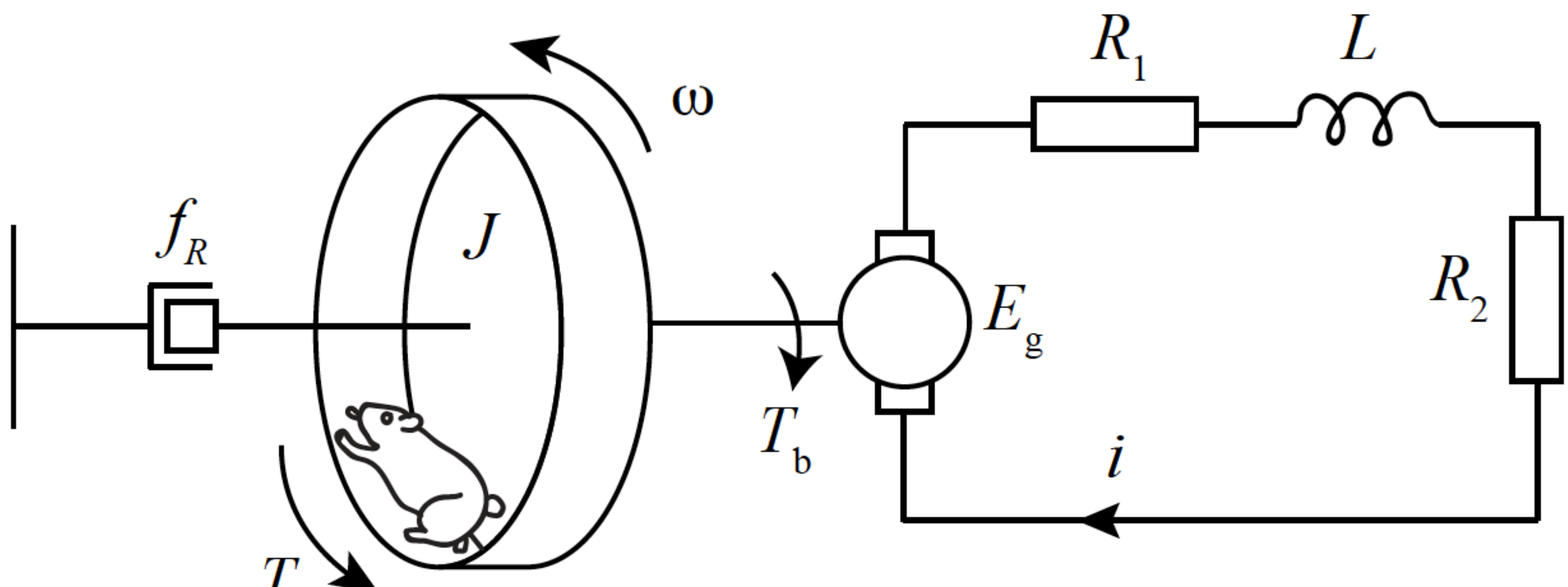
$$\tau_c\ddot{\omega} + \left(1 + \frac{\tau_c}{\tau_m}\right)\dot{\omega} + \left(\frac{1}{\tau_m} + \frac{K_m^2}{L_m f}\right)\omega = \frac{K_m}{f L_m} u_m$$

$$\tau_c = \frac{J}{f} \qquad \tau_m = \frac{L_m}{R_m}$$

Time constants

Example

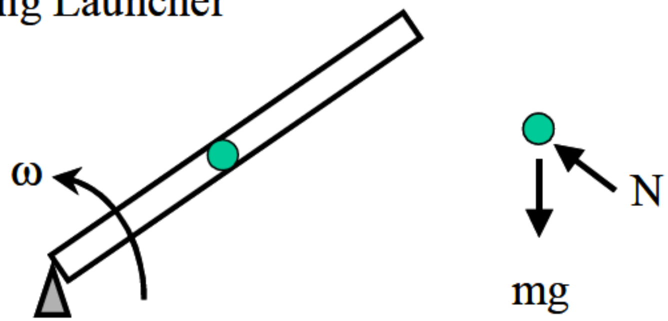
Consider the electromechanical system depicted below. A running rodent provides the input torque T for the electric generator by spinning a wheel. The rotation of the wheel generates voltage E_g that is linearly proportional to the angular velocity ω. A current i starts to flow through the load circuit with resistors of resistances R_1 and R_2 as well as an inductor with inductance L. The current, in return, induces a back-torque denoted by T_b that is linearly proportional to the current and resists the motion of the wheel. The generator and back-torque constants are given by K_g and K_b, respectively. The wheel has an inertia denoted by J while f_R represents the rotational viscous damping coefficient of the shaft.



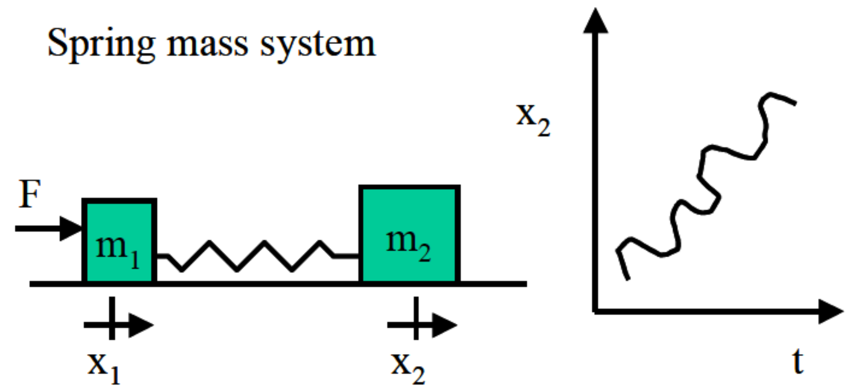
Why Lagrange?

- Newton – Given motion, deduce forces or given forces, solve for motion
- Works very well for simple systems (few variables)

Rotating Launcher



Spring mass system



- Real systems are complex (many variables)
  - Vectoral equations are difficult to manage
  - Constraints are hard to incorporate

Lagrangian Mechanics: Big Picture

- Use kinetic and potential energy to solve for the motion
- From physical vector space to configuration space (scalars)
- No need to solve for accelerations
- Streamlined procedure

- Newton (F = ma) and Lagrangian methods produce the same equations!!

Euler-Lagrange Equations of Motion

- Definition of the Lagrangian

$$L \stackrel{\text{def}}{=} T - V \text{ (Kinetic Energy} - \text{Potential Energy)}$$

- Euler-Lagrange Equation (or equation of motion) for a single coordinate x

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

- The general form of Euler-Lagrange Equation for independent generalized coordinates q_i

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

How to choose generalized coordinates?

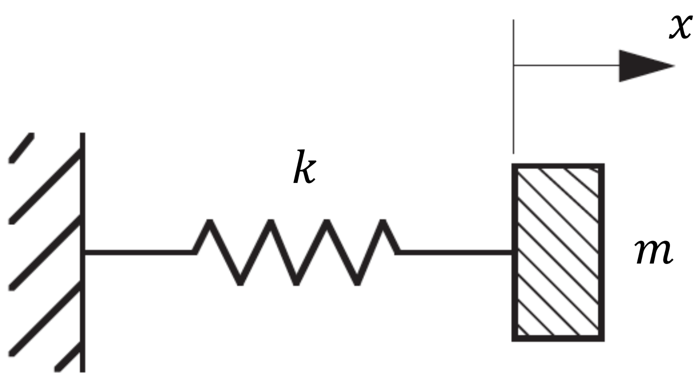
- The choice of coordinates must be independent and orthogonal

- Examples include
  - Cartesian – x, y, z
  - Cylindrical – r, θ, z
  - Spherical – r, θ, ϕ

- The coordinates must locate the body with respect to an inertial reference frame.

- Reminder: An inertial reference frame is one which is not accelerating.

Example: Mass-Spring System

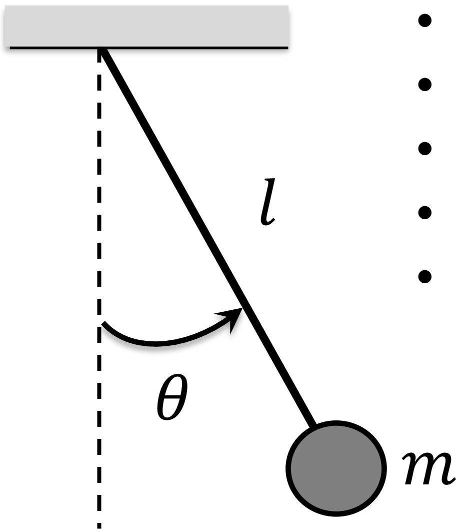


$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$m\ddot{x} = -kx$$

Example: Simple Pendulum



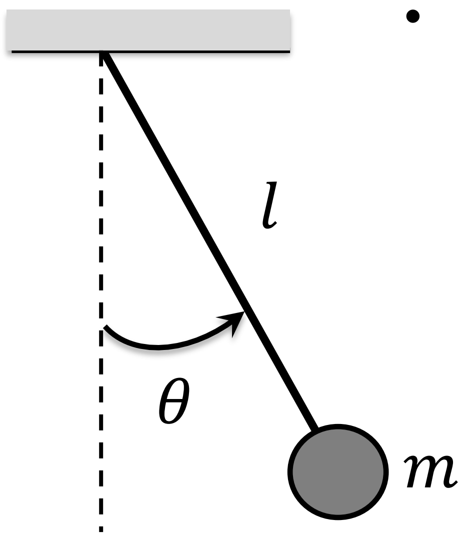
- A pendulum made of a rod with a mass m on the end
- The length of the rod is l
- Assume that the motion takes place in a vertical plane
- Take the pivot point as datum
- Find equations of motion for the generalized coordinate θ

$$T = \frac{1}{2}m(\omega l)^2 = \frac{1}{2}ml^2\dot{\theta}^2 \qquad V = -mgl\cos\theta$$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \quad \rightarrow \quad \ddot{\theta} = -\frac{g}{l}\sin\theta$$

Example: Simple Pendulum



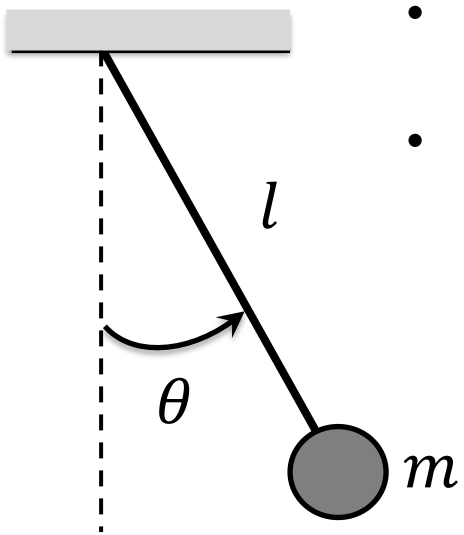
- Kinetic energy calculated using Cartesian coordinates

$$x = l\sin(\theta) \qquad \dot{x} = l\cos(\theta)\,\dot{\theta}$$

$$y = l - l\cos(\theta) \qquad \dot{y} = l\sin(\theta)\,\dot{\theta}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2\dot{\theta}^2$$

Example: Simple Pendulum



- Newton's law gives two equations.

- The equation of motion determining the evolution of θ

$$ml\ddot{\theta} = -mg\sin\theta \quad \rightarrow \quad \ddot{\theta} = -\frac{g}{l}\sin\theta$$

- The equation that determines the reaction force

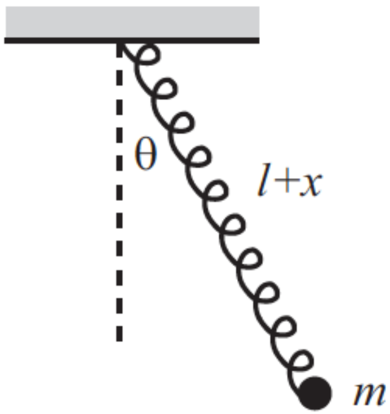
$$F_R = m\left(l\dot{\theta}^2 + g\cos\theta\right)$$

Newton vs Lagrange Formulation

- The Euler-Lagrange method requires **scalar** quantities. No need to perform vector rotations

- The Euler-Lagrange method does not make an explicit reference to the equilibrium reaction forces
  - Disadvantage if we care about them: e.g. choosing a sufficiently strong rope
  - Can be calculated using **Lagrange multipliers**

Example: Spring Pendulum



- A pendulum made of a spring with a mass m on the end
- The equilibrium length of the spring is l
- Assume that the motion takes place in a vertical plane
- Find equations of motion for generalized coordinates x and θ

$$T = \frac{1}{2}m\left(\dot{x}^2 + (l+x)^2\dot{\theta}^2\right)$$

$$V = -mg(l+x)\cos\theta + \frac{1}{2}kx^2$$

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + (l+x)^2\dot{\theta}^2\right) + mg(l+x)\cos\theta - \frac{1}{2}kx^2$$

Example: Spring Pendulum

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + (l+x)^2\dot{\theta}^2\right) + mg(l+x)\cos\theta - \frac{1}{2}kx^2$$

- Note that there are two generalized coordinates, x and θ.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \rightarrow m\ddot{x} = m(l+x)\dot{\theta}^2 + mg\cos\theta - kx \quad (1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \rightarrow \frac{d}{dt}\left(m(l+x)^2\dot{\theta}\right) = -mg(l+x)\sin\theta$$

$$m(l+x)^2\ddot{\theta} + 2m(l+x)\dot{x}\dot{\theta} = -mg(l+x)\sin\theta$$

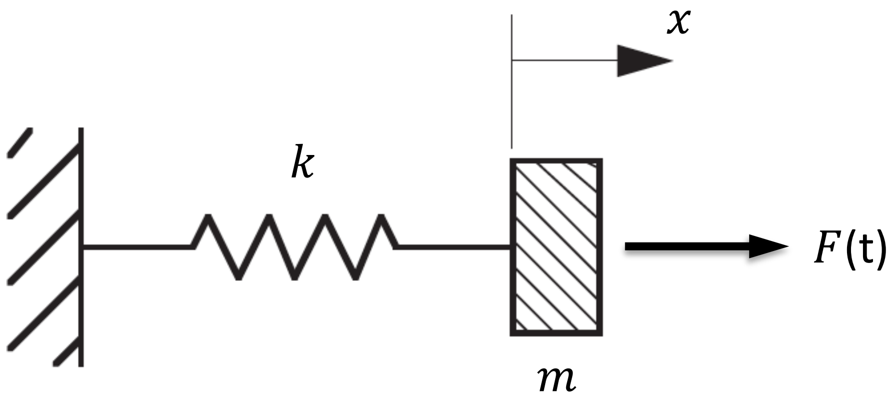
$$m(l+x)\ddot{\theta} + 2m\dot{x}\dot{\theta} = -mg\sin\theta \qquad (2)$$

How to handle external forces?

- Non-conservative virtual work
  - Forces that cannot be derived from a potential function V
  - Externally applied forces, Q_i, fall into this category

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

Example: Force driven spring-mass system



$$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F$$

$$m\ddot{x} = F - kx$$

Rayleigh's Dissipation Function

- The potential function for viscous forces is called Rayleigh dissipation function (**has no physical meaning, only works for linear damping**)

- The Rayleigh dissipation function for a single linear damper is given by

$$D = \frac{1}{2} f \dot{x}^2$$

where f is the damping coefficient and x is the displacement from inertial ground

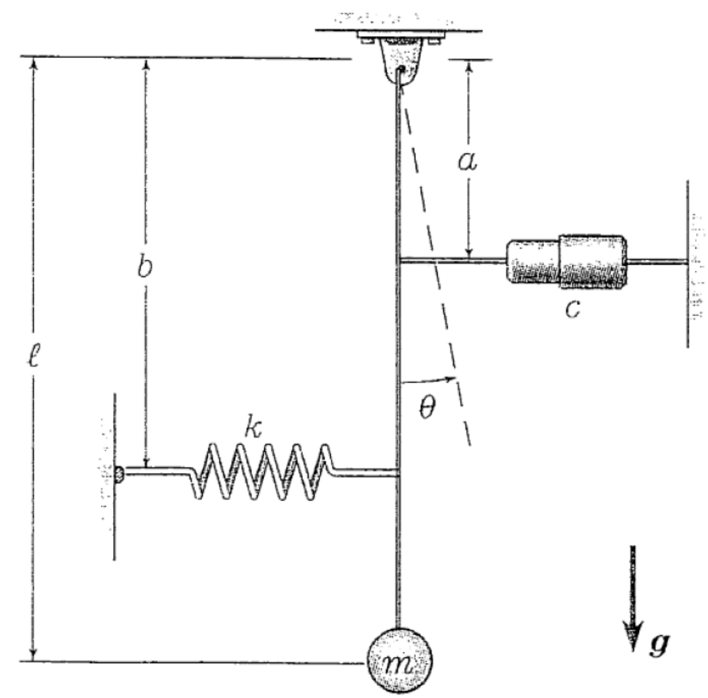
- The most complete form of Lagrange's Equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i$$

Example: Spring-mass-damper system

- A planar pendulum with length l and mass m is restrained by a linear spring of spring constant k and a linear damper of damping coefficient c is shown on the right. The upper end of the rigid and massless link is supported by a frictionless joint.

- Derive the equations of motion for the generalized coordinate θ.



Example: Spring-mass-damper system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \qquad q = \theta$$

$$T = \frac{1}{2}m\left(l\dot{\theta}\right)^2 \qquad V = -mgl\cos(\theta) + \frac{1}{2}k(b\theta)^2 \qquad D = \frac{1}{2}c\left(a\dot{\theta}\right)^2$$

$$L = T - V = \frac{1}{2}m\left(l\dot{\theta}\right)^2 + mgl\cos(\theta) - \frac{1}{2}k(b\theta)^2$$

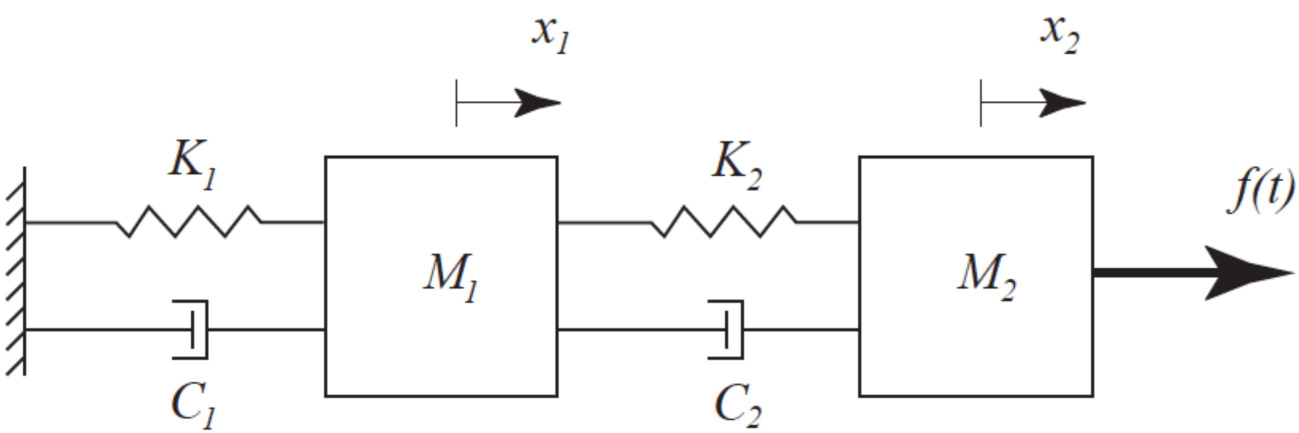
Equation of motion:

$$ml^2\ddot{\theta} + ca^2\dot{\theta} + mgl\sin(\theta) + kb^2\theta = 0$$

For small θ, we can linearize this equation as

$$\sin(\theta) \approx \theta \quad \Rightarrow \quad ml^2\ddot{\theta} + ca^2\dot{\theta} + mgl\,\theta + kb^2\theta = 0$$

Example: Spring-mass-damper system



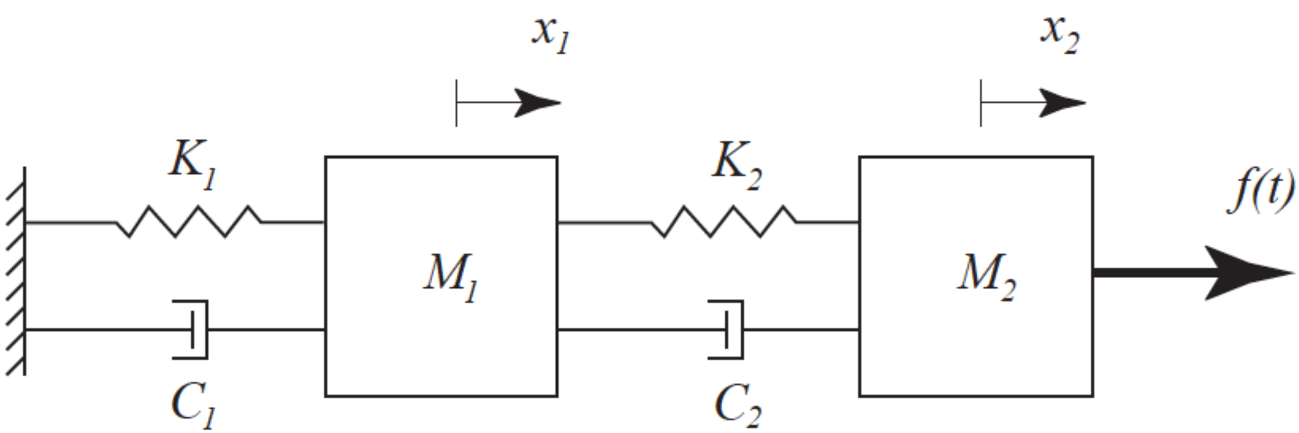
$q_1 = x_1$

$q_2 = x_2$

$$T = \frac{1}{2}\left(M_1 \dot{x}_1{}^2 + M_2 \dot{x}_2{}^2\right) \qquad V = \frac{1}{2}[K_1 x_1{}^2 + K_2(x_2 - x_1)^2]$$

$$D = \frac{1}{2}\left[C_1 \dot{x}_1{}^2 + C_2(\dot{x}_2 - \dot{x}_1)^2\right] \qquad Q_2 = f(t)$$

Example: Spring-mass-damper system



$q_1 = x_1$

$q_2 = x_2$

$$M_1\ddot{x}_1 + (C_1 + C_2)\dot{x}_1 - C_2\dot{x}_2 + (K_1 + K_2)x_1 - K_2x_2 = 0$$

$$M_2\ddot{x}_2 - C_2\dot{x}_1 + C_2\dot{x}_2 - K_2x_1 + K_2\, x_2 = f(t)$$

Principle of Stationary Action

Consider the quantity,

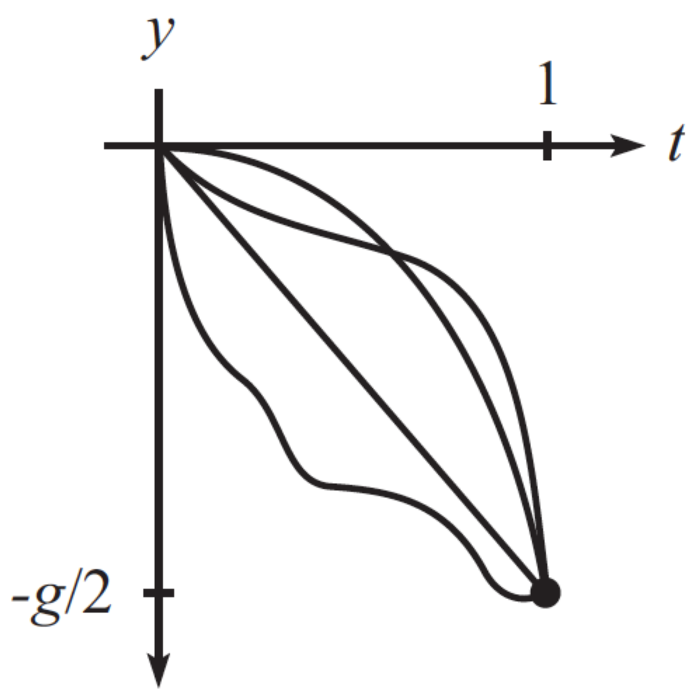
$$S \equiv \int_{t_1}^{t_2} L(x, \dot{x}, t)dt$$

- S is called the action. It is a functional with dimensions of (Energy) x (Time).
- S can be thought of as a function with an infinite number of values, namely all the $\mathrm{x}(t)$ ranging from t_1 to t_2.
- Consider a function $\mathrm{x}(t)$ with its end points fixed, that is $\mathrm{x}(t_1) = x_1$ and $\mathrm{x}(t_2) = x_2$, where x_1 and x_2 are given.
- What function $\mathrm{x}(t)$ yields a stationary value of S? A stationary value is a local minimum, maximum, or saddle point.

Principle of Stationary Action

For example, consider a ball dropped from rest, and consider the function $y(t)$ for $0 \leq t \leq 1$. Assume that we know that $y(0) = 0$ and $y(1) = -g/2$.

Which function shown below would generate a stationary value for S?



Principle of Stationary Action

If the function $x_0(t)$ yields a stationary value (that is a local minimum, maximum, or saddle point) of S, then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_0}\right) = \frac{\partial L}{\partial x_0}$$

Note that we are considering the class of functions whose endpoints are fixed.

Hamilton's principle

The path of a particle is the one that yields a stationary value of the action

Remark: Physical systems mostly act in a way to produce the least action.

Additional Concepts

- Forces of constraints, degrees of freedom
- Calculus of variations
- Virtual Work and D'Alembert's Principle
- Conservation laws and symmetry
- **Noether's Theorem**: For each symmetry of the Lagrangian, there is a conserved quantity.

Newton's vs Lagrange's Methods: Summary Table

Newton's (Direct Approach)	Lagrange's (Indirect Approach)
Accelerations required	Velocities required
Generally vectors required	Generally scalars required
Free-body Diagrams useful	Free-body diagrams not useful
All forces considered	Workless forces (constraints) forces not considered
All forces handled via same expression	Conservative and non-conservative forces handled separately
Intermediate forces more readily available	Intermediate forces less readily available